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New $(p, q; \mu, \nu, f)$-deformed states

M N Hounkonnou and E B Ngompe Nkouankam

International Chair in Mathematical Physics and Applications (ICMPA-UNESCO Chair)
072 B.P.: 50 Cotonou, Republic of Benin

E-mail: norbert_hounkonnou@cipma.net and hounkonnou@yahoo.fr

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Abstract
We construct a new family of deformed states. These states are normalizable on the whole complex plane and continuous in their label z. Under fixed values of parameters, they allow the resolution of unity in the form of an ordinary integral with a positive weight function obtained through the analytic solution of the associated Stieltjes power-moment problem. In addition to the mathematical characteristics, the quantum statistical properties of these states are analytically and numerically discussed in detail in the context of conventional as well as deformed quantum optics. We find that, for non-deformed photons, the states exhibit quadrature squeezing and their photon number statistics is sub-Poissonian. On the other hand, for deformed photons, the states are super-Poissonian and no quadrature squeezing occurs.

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1. Introduction

One of the most fruitful directions in Mathematical Physics from the last few decades of the twentieth century to be presented is related to various deformations of the Heisenberg canonical commutation relation and others which give rise to various types of deformed states, including deformed coherent states. The latter reduce to well-known coherent states for appropriate limit values of deformation parameters. Coherent states are of most importance in several areas of theoretical physics, ranging from quantum optics to statistical mechanics and quantum field theory [1–10, 12–14]. These states, which emerge from the study of the quantum harmonic oscillator, were introduced in the early years of quantum mechanics [8] as wave packets whose dynamics resembles that of a classical particle in a quadratic potential. In modern language [10], the conventional coherent states (CS) $|z\rangle$ of a bosonic harmonic oscillator are the specific superposition of eigenstates $|n\rangle$ of the number operator $N = a^\dagger a$ (with $[a, a^\dagger] = 1$) and parameterized by a single complex variable z. These states may be constructed in three equivalent ways: (i) by defining them as eigenstates of the annihilation operator a, (ii) by applying a unitary displacement operator on the vacuum state $|0\rangle$ (such that $a|0\rangle = 0$) and (iii) by considering them as quantum states with a minimum uncertainty relationship [11].

To arrive at various families of states, which will be nonclassical, it is enough to make slight modifications in each definition of the conventional CS mentioned above. For this reason, many classes of states, which are labelled nowadays as nonclassical, appeared in the literature as some kinds of generalized CS. Such states exhibit some purely quantum-mechanical properties, such as squeezing, higher order squeezing, antibunching, sub-Poissonian statistics, etc (for a review, see [15] and references therein) and thus possess latent applications in optical communication and in precision and sensitive measurements [16]. The generalization of the conventional CS is essentially based on symmetry considerations, dynamics and algebraic aspects. Generalization based on symmetry considerations has led to defining CS for arbitrary classical Lie groups [9]. It should be emphasized that for any given quantum state to be mathematically well defined, it should satisfy a minimum set of conditions [10]: (i) normalizability (as any vector of Hilbert space), (ii) continuity in the label z and (iii) existence of a resolution of unity with a positive-definite weight function (implying that the states form an overcomplete set). If the first two conditions are rather easy to fulfil, the third one, in contrast, imposes some severe restrictions on possible generalized CS. Determining the existence of a unity resolution is indeed a difficult task, which has not been solved for many of the putative CS known in the literature. Recent progress in this domain has been achieved by using some properties of the inverse Mellin and Fourier transform [17].

Since the construction of q-deformed CS associated with the so-called maths-type [18] and physics-type [19, 20] q-deformed boson oscillators [21] as well as (p, q)-deformed CS [22] associated with the (p, q)-deformed boson oscillator, only few of them have been endowed with an explicit form of weight function while dealing with the overcompleteness property. For those connected with maths-type and physics-type q-deformed bosons, the unity resolution has been written as a q-integral [23] with a weight function expressed in terms of a q-exponential [24]. An alternative formulation has been proposed in terms of an ordinary integral, but the corresponding weight function is then only known through the inverse Fourier transform of a given function [25]. Another type of q-deformed CS has also been shown to admit a unity resolution in the form of an ordinary integral with a weight function expressed as a Laplace transform of some given function [26].

More recently, Appl and Schiller [27] introduced a large class of holomorphic quantum states through normalization functions given by generalized hypergeometric functions. This formulation opens new perspectives in the way of generalizing existing deformed states, in particular in terms of bibasic hypergeometric functions. In such a direction, results of previous works [28, 29] as well as the progress achieved by using some properties of the inverse Mellin transform [30] could be of some interest. However these issues, beyond the scope of this work, deserve a proper study. Globally, deformations can be generalized in many ways, involving only three operators: the annihilation, the creation and the number operators (respectively denoted by $a, a^\dagger, N$) with an arbitrary commutation role for a and $a^\dagger$. See [31] for an analysis of relevant physical properties and [32] for mathematical aspects.

The aim of this paper is to provide a generalization of the q-deformed states. Here, the proposed generalization case involves five parameters (p, q, μ, ν and a given function f of p and q). The resolution of unity fulfils the properties we have just mentioned if $\mu = 1$ and $\nu = 0$, corresponding after a rescaling to the case introduced by Quesne [33]. In this case, the generalized CS will be shown to possess some interesting nonclassical properties. It is worth noting that multi-parameter deformation is more flexible when we are dealing with applications of concrete physical models.

The paper is organized as follows. In section 2, the generalized deformed physical states are defined and their properties studied. In section 3, their description in terms of generalized deformed bosons is considered. Geometrical and physical properties in quantum optics are studied in section 4. Finally, the paper ends with some concluding remarks in section 5.

2. New $(p, q; \mu, \nu, f(p, q))$-deformed states in terms of boson operators

As a matter of convenience, let us first briefly recall some deformed oscillator algebras. Related to this, the most commonly used algebras can be classified as follows.

(i) Maths-type q-bosonic algebra. The standard q-deformation, introduced by Arik and Coon, of the harmonic oscillator is generated by the mutually Hermitian conjugate operators a and $a^\dagger$ and the identity operator I satisfying the relation [18]

$$aa^\dagger - qa^\dagger a = I. \tag{1}$$

The associated q-deformed Fock–Hilbert space representation is spanned by the vacuum state $|0\rangle$, annihilated by a, and the orthonormalized states $|n\rangle$, such that

$$a|n\rangle = \sqrt{[n]_q^M}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{[n+1]_q^M}|n+1\rangle, \tag{2}$$

where $[n]_q^M = (1-q^n)/(1-q)$ is the q-basic number. The q-boson coherent states are defined as

$$|z\rangle = \left[\exp_q^M(|z|^2)\right]^{-1/2} \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{[n]_q^M!}}|n\rangle, \tag{3}$$

where $z \in \mathbb{C}$ and

$$\exp_q^M(x) = \sum_{n=0}^{+\infty} \frac{x^n}{[n]_q^M!}, \qquad [n]_q^M! \equiv [n]_q^M\,[n-1]_q^M \ldots [1]_q^M\,. \tag{4}$$

These latter satisfy, for $q > 1$, the required mathematical properties, namely, the normalizability, the continuity, and allow a resolution of unity in terms of an ordinary integral with a simple positive-definite weight function [34]. For $0 < q < 1$, such q-deformed CS were shown [18] to provide a unity resolution written as a q-integral with a weight function expressed in terms of some q-exponential. An alternative formulation [25] was proposed in terms of an ordinary integral with the corresponding weight function known through the formal inverse Fourier transform of some given function.

(ii) Physics-type q-bosonic algebra. The q-deformation, introduced by Biedenharn [19] and Macfarlane [20] in 1989, of the harmonic oscillator is defined by the following relation:

$$aa^\dagger - qa^\dagger a = q^{-N}, \tag{5}$$

N being the number operator. This deformation was a means of obtaining realizations of quantum algebras such as $SU_q(2)$ which arose from physics considerations. Note that a deformation scheme similar to the above proposal was introduced in the work by Sun and Fu [35].

The associated q-deformed Fock–Hilbert space representation is spanned by the vacuum state $|0\rangle$, annihilated by a, and the orthonormalized states $|n\rangle$, such that

$$a|n\rangle = \sqrt{[n]_q^P}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{[n+1]_q^P}|n+1\rangle, \tag{6}$$

where $[n]_q^P = (q^n - q^{-n})/(q - q^{-1})$ is the q-basic number. The q-boson coherent states are defined as

$$|z\rangle = \left[\exp_q^P(|z|^2)\right]^{-1/2} \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{[n]_q^P!}} |n\rangle, \tag{7}$$

where $z \in \mathbb{C}$ and

$$\exp_q^P(x) = \sum_{n=0}^{+\infty} \frac{x^n}{[n]_q^P!}, \qquad [n]_q^P! \equiv [n]_q^P\,[n-1]_q^P \ldots [1]_q^P\,. \tag{8}$$

Let us mention that the terminology Maths-type CS was first introduced in [36] to contrast with the physics-type of MacFarlane and Biedenharn.

One also deals with another q-deformed bosonic algebra due to Quesne.

(iii) Quesne q-deformed bosonic algebra. Another variant of q-deformed algebra is defined by Quesne in [33]:

$$qaa^\dagger - a^\dagger a = I, \qquad aa^\dagger - a^\dagger a = q^{-N-1}. \tag{9}$$

The operators $a, a^\dagger$ act on the Fock–Hilbert space as

$$a|n\rangle = \sqrt{[n]_q^Q}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{[n+1]_q^Q}|n+1\rangle \tag{10}$$

with $a|0\rangle = 0$ and the q-number $[n]_q^Q = (1 - q^{-n})/(q - 1)$. The q-deformed coherent states are expressed as

$$|z\rangle = [E_q((1-q)q|z|^2)]^{-1/2} \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{[n]_q^Q!}} |n\rangle, \tag{11}$$

labelled by $q \in]0, 1[$, and $z \in \mathbb{C}$. $E_q(z) = \prod_{k=0}^{\infty}(1 + q^k z)$ is one of the q-exponentials introduced by Jackson [23].

(iv) (p, q)-oscillator algebra. A (p, q)-generalization of (1) was introduced by Chakrabarty and Jagannathan in [22], as an associative algebra generated by the operators $I, a, a^\dagger$ and N as follows:

$$aa^\dagger - qa^\dagger a = p^{-N} \tag{12}$$

$$aa^\dagger - p^{-1}a^\dagger a = q^N \tag{13}$$

$$[N, a] = -a, \qquad [N, a^\dagger] = a^\dagger. \tag{14}$$

The associated (p, q)-deformed Fock–Hilbert space representation is spanned by the vacuum $|0\rangle$, annihilated by a, and the orthonormalized states $|n\rangle$, such that

$$a|n\rangle = \sqrt{[n]_{p,q}}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{[n+1]_{p,q}}|n+1\rangle, \tag{15}$$

where $[n]_{p,q} = (p^{-n} - q^n)/(p^{-1} - q)$ is called the (p, q)-basic number. The limit $p \to q$ yields the q-oscillator algebra (5). The (p, q)-coherent states corresponding to (12)–(14) can be written in the form

$$|z\rangle = [\mathcal{N}_{p,q}(|z|^2)]^{-1/2} \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{[n]_{p,q}!}} |n\rangle. \tag{16}$$

(v) A (p, q)-generalization of q-Quesne algebras. Let us now introduce a (p, q)-generalization of the above Quesne deformed bosonic algebra (9) as follows:

$$p^{-1}aa^{\dagger} - a^{\dagger}a = q^{-N-1} \tag{17}$$

$$qaa^{\dagger} - a^{\dagger}a = p^{N+1} \tag{18}$$

$$[N, a] = -a \qquad [N, a^{\dagger}] = a^{\dagger}. \tag{19}$$

The associated (p, q)-deformed Fock–Hilbert space is such that

$$a|0\rangle = 0, \qquad a|n\rangle = \sqrt{[n]^{Q}_{p,q}}|n-1\rangle, \qquad a^{\dagger}|n\rangle = \sqrt{[n+1]^{Q}_{p,q}}|n+1\rangle, \tag{20}$$

where $[n]^{Q}_{p,q} = (p^{n} - q^{-n})/(q - p^{-1})$. The Quesne q-algebra is easily recovered in the limit case $p = 1$. The coherent states corresponding to (17) and (18) can be expressed in the form

$$|z\rangle = [\mathcal{M}_{p,q}(|z|^{2})]^{-1/2} \sum_{n=0}^{+\infty} \frac{z^{n}}{\sqrt{[n]^{Q}_{p,q}!}}|n\rangle. \tag{21}$$

In the same vein, let us provide further generalization of the family of harmonic oscillator physical states defined as

$$|z\rangle^{\mu,\nu}_{p,q,f(p,q)} = \left[\mathcal{N}^{\mu,\nu}_{p,q,f}(|z|^{2})\right]^{-1/2} \sum_{n=0}^{+\infty} \frac{z^{n}}{\sqrt{[n]^{\mu,\nu}_{p,q,f}!}}|n\rangle, \tag{22}$$

where

$$\begin{aligned} [n]^{\mu,\nu}_{p,q,f}! &\equiv [n]^{\mu,\nu}_{p,q,f}[n-1]^{\mu,\nu}_{p,q,f}\ldots[2]^{\mu,\nu}_{p,q,f}[1]^{\mu,\nu}_{p,q,f} \quad \text{if} \quad n = 1, 2, \ldots \\ &\equiv 1 \quad \text{if} \quad n = 0 \end{aligned} \tag{23}$$

and

$$[n]^{\mu,\nu}_{p,q,f} \equiv f(p, q)\frac{q^{\nu n}}{p^{\mu n}}\left(\frac{p^{n} - q^{-n}}{q - p^{-1}}\right). \tag{24}$$

Here $|n\rangle = (n!)^{-1/2}(a^{\dagger})^{n}|0\rangle$ is an n-boson state. a and $a^{\dagger}$ being, as in the following, the annihilation and the creation operators, of the conventional (non-deformed) boson oscillator ($[a, a^{\dagger}] = 1$), respectively. Let us recall that the number operator of the non-deformed harmonic oscillator is given by $N = a^{\dagger}a$.

It is worth noting that the deformed number (24) generalizes all the deformed algebras mentioned above. Indeed, we have the following relations:

$$(i) \quad [n]^{0,0}_{p,q,1} = [n]^{Q}_{p,q} \qquad (ii) \quad [n]^{1,1}_{p,q,1} = [n]_{p,q} \tag{25}$$

$$(iii) \quad [n]^{1,0}_{p,q,\frac{1}{p}} = [n]^{Q}_{pq} \qquad (iv) \quad [n]^{0,1}_{p,q,\frac{1}{p}} = [n]^{M}_{pq} \tag{26}$$

$$(v) \quad [n]^{\mu,\mu}_{q,q,1} = [n]^{P}_{q}\,. \tag{27}$$

Although the multiparameter deformation remains to understand from the physical point of view, it is useful to build a general framework of analysis for various types of deformations raised in the literature on a unified mathematical basis. Indeed, in particular cases, the functions $f(p, q)$ and the parameters μ and ν lead to known deformations with corresponding deformed numbers. See, for instance, (25)–(27).

In general, the parameters p and q may be real or a phase factor, but we require, throughout, p and q to be real and positive such that $0 < pq < 1$, $p^{\mu} < q^{\nu-1}$, $p > 1$. f is a well behaved real and non-negative function of deformation parameters p and q, satisfying

$$\lim_{(p,q)\to(1,1)} f(p,q) = 1. \tag{28}$$

As an illustration, the following functions:

$$(i) \quad f(p,q) = 1 \qquad (ii) \quad f(p,q) = \exp(p^2 - q^2) \tag{29}$$

$$(iii) \quad f(p,q) = \frac{p^2}{2 - pq} \qquad (iv) \quad f(p,q) = p - q + 1, \tag{30}$$

satisfying the fact that the above-mentioned requirements could be taken as good candidates for $f(p,q)$.

It is noteworthy that if $\nu = 0$ and $f(p,q) \equiv 1$ then, in the limit $p \to 1$, $[n]^{\mu,0}_{p,q,1}$ reduces to $[n]_q = (1 - q^{-n})/(q-1)$. The $(p, q; \mu, \nu, f(p,q))$-number can be expressed as

$$[n]^{\mu,\nu}_{p,q,f} = f(p,q)\left(\frac{q^{\nu-1}}{p^{\mu-1}}\right)^n [n]_{p,q}. \tag{31}$$

For $(p,q) \to (1,1)$, $[n]^{\mu,\nu}_{p,q,f}$ and $[n]^{\mu,\nu}_{p,q,f}!$ go to n and $n!$, respectively.

Let us now investigate some properties of the states (22).

2.1. Normalizability

The normalization condition of (22), ${}^{\mu,\nu}_{p,q,f(p,q)}\langle z|z\rangle^{\mu,\nu}_{p,q,f(p,q)} = 1$, leads to

$$\mathcal{N}^{\mu,\nu}_{p,q,f}(x) = \sum_{n=0}^{+\infty} \frac{x^n}{[n]^{\mu,\nu}_{p,q,f}!}, \tag{32}$$

where $x = |z|^2$.

Let

$$a_n = \frac{1}{[n]^{\mu,\nu}_{p,q,f}!}.$$

Then

$$\lim_{n\to+\infty} \frac{a_n}{a_{n-1}} = \lim_{n\to+\infty}\left[f(p,q)\left(\frac{q^{\nu-1}}{p^{\mu}}\right)^n\left(\frac{p^n q^n - 1}{q - p^{-1}}\right)\right]^{-1} = 0 \tag{33}$$

since $q^{\nu-1}/p^{\mu} > 1$ and $0 < pq < 1$, proving that the radius of convergence of (32) is $R = +\infty$. The rhs of (32) is an entire function. Moreover, it is positive in the real axis. The states (22) are therefore normalizable on the whole complex plane.

For $\nu = \mu = 0$ and $f(p,q) \equiv 1$, (32) can be expressed in terms of the (p,q)-analogue of the exponential defined by Floreanini *et al* [28], $E_{p,q}$, namely

$$\mathcal{N}^{0,0}_{p,q,1}(x) = E_{p,q}\left(q\left(1 - \frac{q}{p}\right)x\right). \tag{34}$$

For $\nu = 0$ and $\mu = 1$, (32) writes

$$\mathcal{N}^{1,0}_{p,q,f}(x) = \sum_{n=0}^{+\infty} q^{\frac{n(n-1)}{2}} \frac{\left(\frac{p^{-1}-q}{f(p,q)}x\right)^n}{[p,q;p,q]_n}, \tag{35}$$

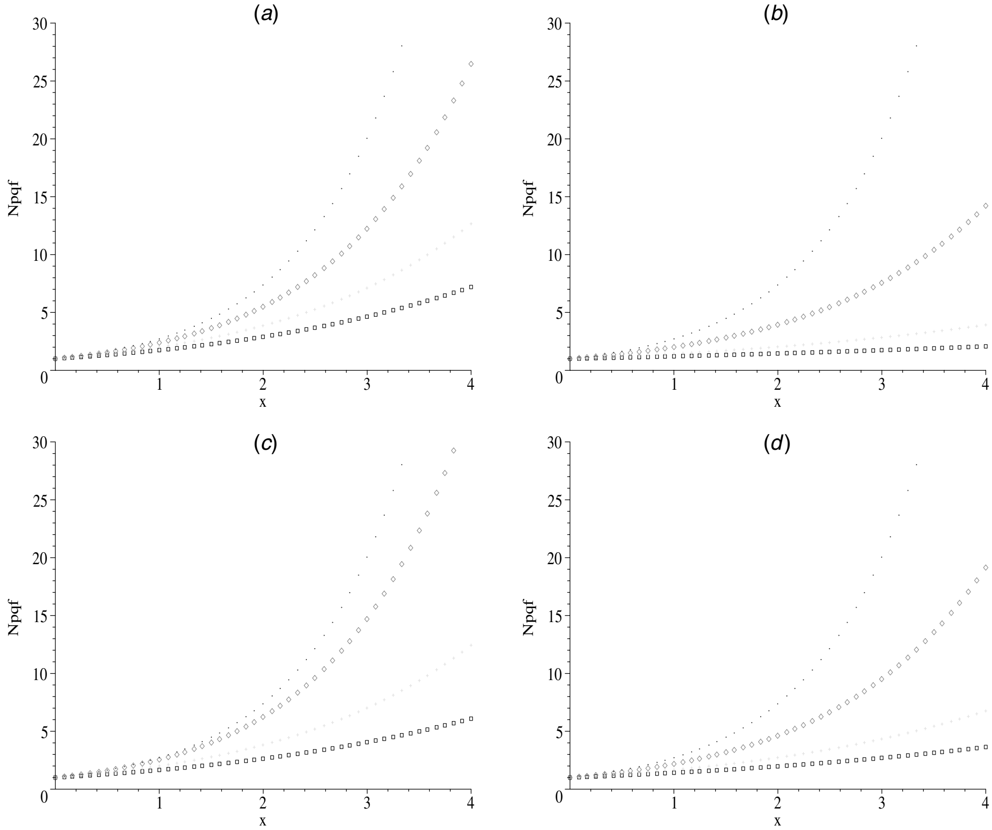

Figure 1. The normalization function $\mathcal{N}^{1,0}_{p,q,f}(x)$ versus x for various (p, q) values: $(p, q) = (1, 1)$ (pointed line), $(p, q) = (1.01, 0.9)$ (diamond line), $(p, q) = (1.1, 0.8)$ (crossed line), $(p, q) = (1.2, 0.7)$ (boxed line).

where

$$[p, q; p, q]_n = \left(\frac{1}{p^n} - q^n\right)\left(\frac{1}{p^{n-1}} - q^{n-1}\right)\cdots\left(\frac{1}{p} - q\right). \tag{36}$$

For $(p, q) \to (1, 1)$, the rhs of (35) reduces to $\mathcal{N}(x) = e^x$, as is shown in figure 1, where, as in the following, the graphs (a), (b), (c) and (d) correspond, respectively, to the cases $f(p, q) = 1$, $f(p, q) = \exp(p^2 - q^2)$, $f(p, q) = p^2/(2 - pq)$ and $f(p, q) = p - q + 1$.

2.2. Continuity in z

The following statement is readily satisfied.

Proposition 2.1. *The states* $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$, *defined in (22), are continuous in* z.

Proof. By definition, (22) are continuous in z if

$$|z - z'| \to 0 \Rightarrow \||z\rangle^{\mu,\nu}_{p,q,f(p,q)} - |z'\rangle^{\mu,\nu}_{p,q,f(p,q)}|^2 \to 0. \tag{37}$$

We get

$$\||z\rangle^{\mu,\nu}_{p,q,f(p,q)} - |z'\rangle^{\mu,\nu}_{p,q,f(p,q)}|^2 = 2\big(1 - Re\big(^{\mu,\nu}_{p,q,f(p,q)}\langle z|z'\rangle^{\mu,\nu}_{p,q,f(p,q)}\big)\big), \tag{38}$$

where

$$ {}_{p,q,f(p,q)}^{\mu,\nu}\langle z|z'\rangle_{p,q,f(p,q)}^{\mu,\nu} = \left[\mathcal{N}_{p,q,f}^{\mu,\nu}(|z|^2)\mathcal{N}_{p,q,f}^{\mu,\nu}(|z'|^2)\right]^{-1/2} \sum_{n=0}^{+\infty} \frac{z'^{*n} z^n}{[n]_{p,q,f}^{\mu,\nu}!}. \qquad (39) $$

If $z' \to z$, then ${}_{p,q,f(p,q)}^{\mu,\nu}\langle z|z'\rangle_{p,q,f(p,q)}^{\mu,\nu} \to 1$. Hence, condition (37) is satisfied. □

2.3. Resolution of unity

To investigate the resolution of unity of the states (22), we assume the existence of a positive weight $W_{p,q,f}^{\mu,\nu}(|z|^2)$ such that the resolution of the identity reads

$$ \int\int_{\mathbb{C}} \mathrm{d}^2 z |z\rangle_{p,q,f(p,q)}^{\mu,\nu} W_{p,q,f}^{\mu,\nu}(|z|^2)_{p,q,f(p,q)}^{\mu,\nu}\langle z| = \sum_{n=0}^{+\infty} |n\rangle\langle n| = I, \qquad (40) $$

where $\mathrm{d}^2 z = \mathrm{d}(\mathrm{Re}\, z)\mathrm{d}(\mathrm{Im}\, z)$.

Substituting $z = r\,\mathrm{e}^{\mathrm{i}\varphi}$ into the lhs of (40) and integrating over φ, the function $W_{p,q,f}^{\mu,\nu}(x)$ is required to be in the form

$$ W_{p,q,f}^{\mu,\nu}(x) = \frac{1}{\pi}\mathcal{N}_{p,q,f}^{\mu,\nu}(x)\tilde{W}_{p,q,f}^{\mu,\nu}(x), \qquad x = r^2, \qquad (41) $$

where $\tilde{W}_{p,q,f}^{\mu,\nu}(x)$ is to be determined from the equation

$$ \int_0^{+\infty} x^n \tilde{W}_{p,q,f}^{\mu,\nu}(x)\,\mathrm{d}x = [n]_{p,q,f}^{\mu,\nu}!, \qquad n = 0, 1, 2, \ldots, \infty. \qquad (42) $$

This is the classical Stieltjes power-moment problem [37].

If n in (42) is extended to $s-1$, where $s \in \mathbb{C}$, then the problem can be formulated in terms of the Mellin and inverse Mellin transforms [38] that have been extensively used in the context of various kinds of generalized coherent states. By setting $\rho_{p,q,f}^{\mu,\nu}(n) = [n]_{p,q,f}^{\mu,\nu}!$, here $\rho_{p,q,f}^{\mu,\nu}(s-1)$ is the Mellin transform, $\mathcal{M}\left[\tilde{W}_{p,q,f}^{\mu,\nu}(x); s\right]$, of $\tilde{W}_{p,q,f}^{\mu,\nu}(x)$, i.e.

$$ \rho_{p,q,f}^{\mu,\nu}(s-1) = \mathcal{M}\left[\tilde{W}_{p,q,f}^{\mu,\nu}(x); s\right] \equiv \int_0^{+\infty} x^{s-1}\tilde{W}_{p,q,f}^{\mu,\nu}(x)\,\mathrm{d}x \qquad (43) $$

and $\tilde{W}_{p,q,f}^{\mu,\nu}(x)$ in turn is the inverse Mellin transform, $\mathcal{M}^{-1}\left[\rho_{p,q,f}^{\mu,\nu}(s-1); x\right]$, of the value $\rho_{p,q,f}^{\mu,\nu}(s-1)$, i.e.

$$ \tilde{W}_{p,q,f}^{\mu,\nu}(x) = \mathcal{M}^{-1}\left[\rho_{p,q,f}^{\mu,\nu}(s-1); x\right] \equiv \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \rho_{p,q,f}^{\mu,\nu}(s-1)x^{-s}\,\mathrm{d}s. \qquad (44) $$

For $\mu = 1$, $\nu = 0$, Quesne [33] shows that the weight function is explicitly given by

$$ W_{p,q,f}^{1,0}(x) = \frac{p^{-1}-q}{\pi f(p,q)\ln\left(\frac{1}{pq}\right)} e_{pq}\left(-\frac{p^{-1}-q}{f(p,q)}x\right)\mathcal{N}_{p,q,f}^{1,0}(x), \qquad (45) $$

where

$$ e_q(x) = \left[\prod_{n=0}^{\infty}(1-xq^n)\right]^{-1}. \qquad (46) $$

Unfortunately, to our best knowledge of existing mathematical tools, the other values of μ and ν (i.e. $\mu \neq 1$ or $\nu \neq 0$) fail to give explicit analytical expressions for the weight functions deducible from non-deformed integrals.

The solution $\tilde{W}_{p,q,f}^{\mu,\nu}(x)$ of the power-moment problem (42) is not unique. According to Calerman's condition [37], our solution would be unique (non-unique) if the sum

$S = \sum_{n=1}^{\infty} a_n$, where $a_n \equiv \left([n]^{\mu,\nu}_{p,q,f}!\right)^{-1/(2n)}$ diverges (converges). We now apply the logarithmic test [39] to examine the convergence of S. The logarithmic criterion stipulates that if $T = \lim_{n\to+\infty}(\ln a_n/\ln n) > -1(< -1)$, then S diverges (converges). Since $p^{\mu}/q^{\nu-1} < 1$, we have $\lim_{n\to+\infty} -n(n+1)\ln(q^{\nu-1}/p^{\mu})/4n\ln n < -1$. This proves the non-uniqueness of the function $\tilde{W}^{\mu,\nu}_{p,q,f}(x)$.

3. Description of the $(p, q; \mu, \nu, f(p, q))$-deformed states in terms of $(p, q; \mu, \nu, f(p, q))$-deformed bosons

The $(p, q; \mu, \nu, f)$-deformed states studied in section 2 in terms of the conventional boson operators a and $a^{\dagger}$ can alternatively be described in terms of the $(p, q; \mu, \nu, f)$-deformed boson operators b and $b^{\dagger}$ defined in terms of the latter by [31, 40]

$$b^{\dagger} = \sqrt{\frac{[N]}{N}}a^{\dagger}, \qquad b = a\sqrt{\frac{[N]}{N}}, \tag{47}$$

where $[N]$ is given by

$$[N] = f(p, q)\left(\frac{q^{\nu}}{p^{\mu}}\right)^{N}\left(\frac{p^{N} - q^{-N}}{q - p^{-1}}\right). \tag{48}$$

In contrast to the boson operators for which $a^{\dagger}a = N$, $aa^{\dagger} = N + 1$, the following statements hold for $(p, q; \mu, \nu, f)$-deformed operators

Proposition 3.1.

$$(i) \quad b^{\dagger}b = [N], \qquad bb^{\dagger} = [N + 1]. \tag{49}$$

$$(ii) \quad \frac{p^{\mu-1}}{q^{\nu-1}}bb^{\dagger} - qb^{\dagger}b = f(p, q)\frac{q^{(\nu-1)N}}{p^{\mu N}}. \tag{50}$$

For $\mu = \nu = 0$ and $f(p, q) \equiv 1$, relation (50) yields (17).

$$(iii) \quad [N, b^{\dagger}] = b^{\dagger}, \qquad [N, b] = -b. \tag{51}$$

Proof. It is immediate from definitions (47) and (48). □

The operators $b^{\dagger}, b$ act on the same Fock space $\{|n\rangle | n = 0, 1, 2, \ldots\}$ as $a^{\dagger}, a$:

$$b|n\rangle = \sqrt{[n]^{\mu,\nu}_{p,q,f}}|n - 1\rangle, \qquad b^{\dagger}|n\rangle = \sqrt{[n + 1]^{\mu,\nu}_{p,q,f}}|n + 1\rangle, \tag{52}$$

with $b|0\rangle = 0$. The n-boson states $|n\rangle$ can, therefore, be rewritten as n-deformed boson states through the relation

$$|n\rangle = \frac{1}{\sqrt{[n]^{\mu,\nu}_{p,q,f}!}}(b^{\dagger})^{n}|0\rangle. \tag{53}$$

On inserting (53) in (22), we can express the $(p, q; \nu, \mu, f)$-deformed states in terms of the $(p, q; \nu, \mu, f)$-deformed boson operators $b^{\dagger}, b$ as

$$|z\rangle^{\mu,\nu}_{p,q,f(p,q)} = \left[\mathcal{N}^{\mu,\nu}_{p,q,f}(|z|^{2})\right]^{-1/2}\sum_{n=0}^{+\infty}\frac{(zb^{\dagger})^{n}}{\sqrt{[n]^{\mu,\nu}_{p,q,f}!}}|0\rangle. \tag{54}$$

From (22) and (54), it follows that

$$b|z\rangle^{\mu,\nu}_{p,q,f(p,q)} = z|z\rangle^{\mu,\nu}_{p,q,f(p,q)}, \tag{55}$$

showing that the $(p, q; \mu, \nu, f)$-deformed states (22) are annihilation operator states for the $(p, q; \mu, \nu, f)$-deformed oscillator algebra generated by $N, b, b^\dagger$, and defined by (*ii*) and (*iii*) of proposition 3.1. In the limit $p, q \to 1$, (54) and (55) reduce to known results for the conventional CS of bosonic oscillators, namely, $|z\rangle = (\zeta(x))^{-1/2}\exp(za^\dagger)|0\rangle$ and $a|z\rangle = z|z\rangle$, where $\zeta(x) = \exp(x)$.

4. Geometrical and quantum statistical properties of the states $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$

In this section, we proceed to study some geometrical and physical properties of the $(p, q, \mu; \nu, f)$-deformed states with applications in the case where the resolution of unity is explicitly solved, namely, for $\mu = 1, \nu = 0$ and for different values of $f(p, q)$ given in (29) and (30). For such a purpose, we shall need to evaluate the expectation values of some monomials in the boson creation and annihilation operators $a^\dagger, a$. These are defined by

$$\langle (a^\dagger)^{p'} a^r\rangle \equiv {}^{\mu,\nu}_{p,q,f(p,q)}\langle z|(a^\dagger)^{p'} a^r|z\rangle^{\mu,\nu}_{p,q,f(p,q)}. \tag{56}$$

More explicitly, from

$$a^r|n\rangle = \sqrt{n(n-1)\ldots(n-r+1)}|n-r\rangle, \qquad 0 \leqslant r \leqslant n \tag{57}$$

and

$$(a^\dagger)^{p'}|n\rangle = \sqrt{(n+1)(n+2)\ldots(n+p')}|n+p'\rangle, \tag{58}$$

we readily obtain

$${}^{\mu,\nu}_{p,q,f(p,q)}\langle z|(a^\dagger)^{p'} a^r|z\rangle^{\mu,\nu}_{p,q,f(p,q)} = \left[\mathcal{N}^{\mu,\nu}_{p,q,f}(|z|^2)\right]^{-1} \sum_{n=r}^{+\infty}\sum_{m=0}^{+\infty} h^{r,p'}_{m,n}\langle m|n+p'-r\rangle, \tag{59}$$

where

$$h^{r,p'}_{m,n} = \frac{\sqrt{n(n-1)\ldots(n-r+1)}\sqrt{(n-r+1)(n-r+2)\ldots(n-r+p')}}{\sqrt{[n]^{\mu,\nu}_{p,q,f}!}\sqrt{[m]^{\mu,\nu}_{p,q,f}!}} z^{*m} z^n. \tag{60}$$

Therefore, (56) can be rewritten as

$$\langle (a^\dagger)^{p'} a^r\rangle = (z^*)^{p'} z^r \mathbf{S}^{(p',r)}_{p,q,\mu,\nu,f}(x), \qquad r = p' = 0, 1, 2, \ldots, \tag{61}$$

where

$$\mathbf{S}^{(p',r)}_{p,q,\mu,\nu,f}(x) = \frac{1}{\mathcal{N}^{\mu,\nu}_{p,q,f}(x)} \sum_{n=0}^{+\infty} \left(\frac{(n+r)!(n+p')!}{[n+r]^{\mu,\nu}_{p,q,f}![n+p']^{\mu,\nu}_{p,q,f}!}\right)^{1/2} \frac{x^n}{n!}. \tag{62}$$

For $r = p'$, (61) can be expressed through the derivatives of $\mathcal{N}^{\mu,\nu}_{p,q,f}(x)$ as [26]

$$\langle (a^\dagger)^r a^r\rangle = \frac{x^r}{\mathcal{N}^{\mu,\nu}_{p,q,f}(x)} \frac{\mathrm{d}^r \mathcal{N}^{\mu,\nu}_{p,q,f}(x)}{\mathrm{d}x^r}. \tag{63}$$

From now on, unless it is necessary, the sub-indices p, q, μ, ν, f will be removed from $\mathbf{S}^{(p',r)}_{p,q,\mu,\nu,f}$ and the indices from $\mathcal{N}^{\mu,\nu}_{p,q,f}$.

4.1. Geometry of the states $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$

The geometry of any quantum state space can be described by the corresponding metric tensor. This real and positive definite metric is defined on the underlying manifold that the quantum states form, or belong to, by calculating the distance function (line element) between two quantum states. So, it is also known as a Fubini–Study metric of the ray space. The knowledge of the quantum metric enables one to calculate quantum mechanical transition probability and uncertainties [41].

The map from z to $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$, which is a map from the space $\mathbb{C}$ of complex numbers onto a continuous subset of unit vectors in Hilbert space, generates in the latter a two-dimensional surface with the following Fubini–Study element:

$$\mathrm{d}\sigma^2_{p,q,\mu,\nu,f} = W^{\mu,\nu}_{p,q,f}(|z|^2)\,\mathrm{d}z^*\,\mathrm{d}z \tag{64}$$

with

$$W^{\mu,\nu}_{p,q,f}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\langle N\rangle = \left(\frac{x\mathcal{N}'(x)}{\mathcal{N}(x)}\right)', \qquad x = |z|^2 \tag{65}$$

as the corresponding metric factor. Here, primes denote the order of differentiation with respect to the variable x. In polar coordinates, $z = r\,\mathrm{e}^{\mathrm{i}\theta}$, it follows that

$$\mathrm{d}\sigma^2_{p,q,\mu,\nu,f} = W^{\mu,\nu}_{p,q,f}(r^2)(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2). \tag{66}$$

The result, therefore, is a circular-symmetric, two-dimensional geometry. In the limit $(p, q) \to (1, 1)$, $W^{\mu,\nu}_{p,q,f}(x) = 1$. In this situation, $\mathrm{d}\sigma^2_{p,q,\mu,\nu,f}$ describes a flat, two-dimensional surface. Otherwise, $W^{\mu,\nu}_{p,q,f}(x) \neq 1$ and the geometry is non-flat.

For $x \ll 1$, we obtain

$$W^{\mu,\nu}_{p,q,f}(x) = \frac{p^{\mu-1}}{f(p,q)q^{\nu-1}}\left\{1 + \frac{2p^{\mu-1}}{f(p,q)q^{\nu-1}}\left[\frac{2p^{\mu}}{q^{\nu-1}(pq+1)} - 1\right]x + o(x^2)\right\}, \tag{67}$$

which reduces for $\mu = 1$, $\nu = 0$ to

$$W^{1,0}_{p,q,f}(x) = \frac{q}{f(p,q)}\left[1 - \frac{2q(1-pq)}{f(p,q)(pq+1)}x + o(x^2)\right]. \tag{68}$$

Figure 2 shows that $W^{1,0}_{p,q,f}(x) < 1$ for the values of $f(p,q)$ under consideration.

4.2. Quantum statistical properties of $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$: conventional photons

Here we investigate quantum statistical properties of the states $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$ with respect to the non-deformed operators a and $a^\dagger$ that describe conventional photons.

4.2.1. Photon number distribution. The probability of finding n quanta in the $(p, q; \mu, \nu, f)$-deformed states $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$, i.e. its photon-number distribution, is given by [26]

$$P^{\mu,\nu}_{p,q,f}(n, x) = \frac{x^n}{\mathcal{N}(x)[N]!}, \tag{69}$$

which reduces to the well-known Poisson distribution for the non-deformed CS in the limit $p, q \to 1$.

Since for the non-deformed coherent states the variance of the number operator is equal to its average, deviations from the Poisson distribution can be measured with the Mandel parameter [42]

$$Q^{\mu,\nu}_{p,q,f}(x) = \frac{(\Delta N)^2 - \langle N\rangle}{\langle N\rangle}, \qquad (\Delta N)^2 = \langle N^2\rangle - \langle N\rangle^2, \tag{70}$$

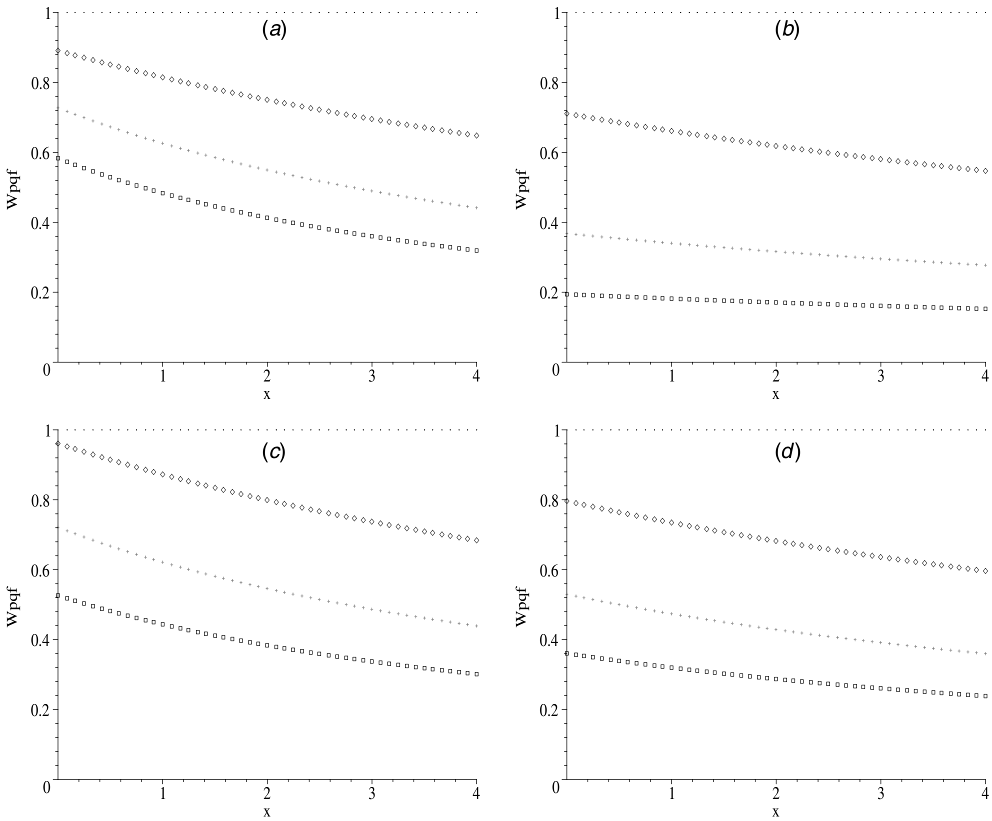

Figure 2. The metric factor $W_{p,q,f}^{1,0}$ versus x or various (p, q) values: $(p, q) = (1, 1)$ (pointed line), $(p, q) = (1.01, 0.9)$ (diamond line), $(p, q) = (1.1, 0.8)$ (crossed line), $(p, q) = (1.2, 0.7)$ (boxed line).

which vanishes for the Poisson distribution, is negative for a sub-Poissonian distribution (photon-antibunching) and positive for a super-Poissonian distribution (photon-bunching). From (63), one readily finds

$$Q_{p,q,f}^{\mu,\nu}(x) = x\left(\frac{\mathcal{N}''(x)}{\mathcal{N}'(x)} - \frac{\mathcal{N}'(x)}{\mathcal{N}(x)}\right). \tag{71}$$

For $x \ll 1$, we obtain

$$Q_{p,q,f}^{\mu,\nu}(x) = \frac{p^{\mu-1}}{q^{\nu-1} f(p,q)}\left[\frac{2p^{\mu}}{q^{\nu-1}(pq+1)} - 1\right] x + o(x^2) \tag{72}$$

reducing, for $\mu = 1$, $\nu = 0$ to

$$Q_{p,q,f}^{1,0}(x) = -\frac{q}{f(p,q)}\left(\frac{1-pq}{1+pq}\right) x + o(x^2). \tag{73}$$

As one can see in figure 3, $Q_{p,q,f}^{1,0}(x) < 0$, which yields a sub-Poissonian distribution.

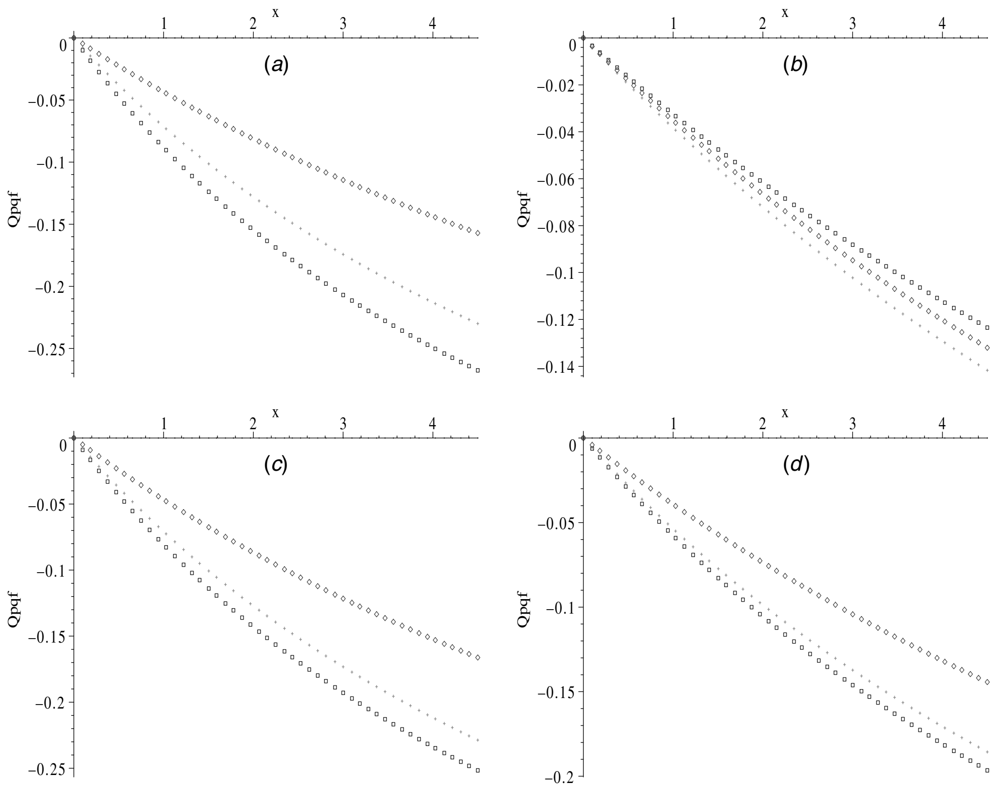

Figure 3. The Mandel parameter $Q^{1,0}_{p,q,f}$ versus x for various (p, q) values: $(p, q) = (1.01, 0.9)$ (diamond line), $(p, q) = (1.1, 0.8)$ (crossed line), $(p, q) = (1.2, 0.7)$ (boxed line).

4.2.2. Squeezing properties. Let us consider the conventional quadrature operators X and P defined in terms of non-deformed operators a and $a^\dagger$:

$$X = \frac{1}{\sqrt{2}}(a + a^\dagger), \qquad P = \frac{1}{\mathrm{i}\sqrt{2}}(a - a^\dagger). \tag{74}$$

The commutation relation for a and $a^\dagger$ leads to the following uncertainty relation:

$$(\Delta X)^2(\Delta P)^2 \geqslant \tfrac{1}{4}|[X, P]|^2 = \tfrac{1}{4}. \tag{75}$$

In the vacuum state $|0\rangle$, we have $(\Delta X)^2_0 = (\Delta P)^2_0 = 1/2$ and so $(\Delta X)^2_0(\Delta P)^2_0 = 1/4$. While it is impossible to lower the product $(\Delta X)^2(\Delta P)^2$ below the vacuum uncertainty value, it is nevertheless possible to define the squeezed states for which at most one quadrature variance lies below the vacuum value,

$$(\Delta X)^2 < \tfrac{1}{2} \qquad \text{or} \qquad (\Delta P)^2 < \tfrac{1}{2}. \tag{76}$$

For the $(p, q; \mu, \nu, f)$-deformed states, it is straightforward to show that the variances of X and P are given by

$$(\Delta X)^2 = 2(\mathrm{Re}(z))^2[\mathbf{S}^{(2,0)}(x) - (\mathbf{S}^{(1,0)}(x))^2] + x[\mathbf{S}^{(1,1)}(x) - \mathbf{S}^{(2,0)}(x)] + \tfrac{1}{2}, \tag{77}$$

$$(\Delta P)^2 = 2(\mathrm{Im}(z))^2[\mathbf{S}^{(2,0)}(x) - (\mathbf{S}^{(1,0)}(x))^2] + x[\mathbf{S}^{(1,1)}(x) - \mathbf{S}^{(2,0)}(x)] + \tfrac{1}{2}, \tag{78}$$

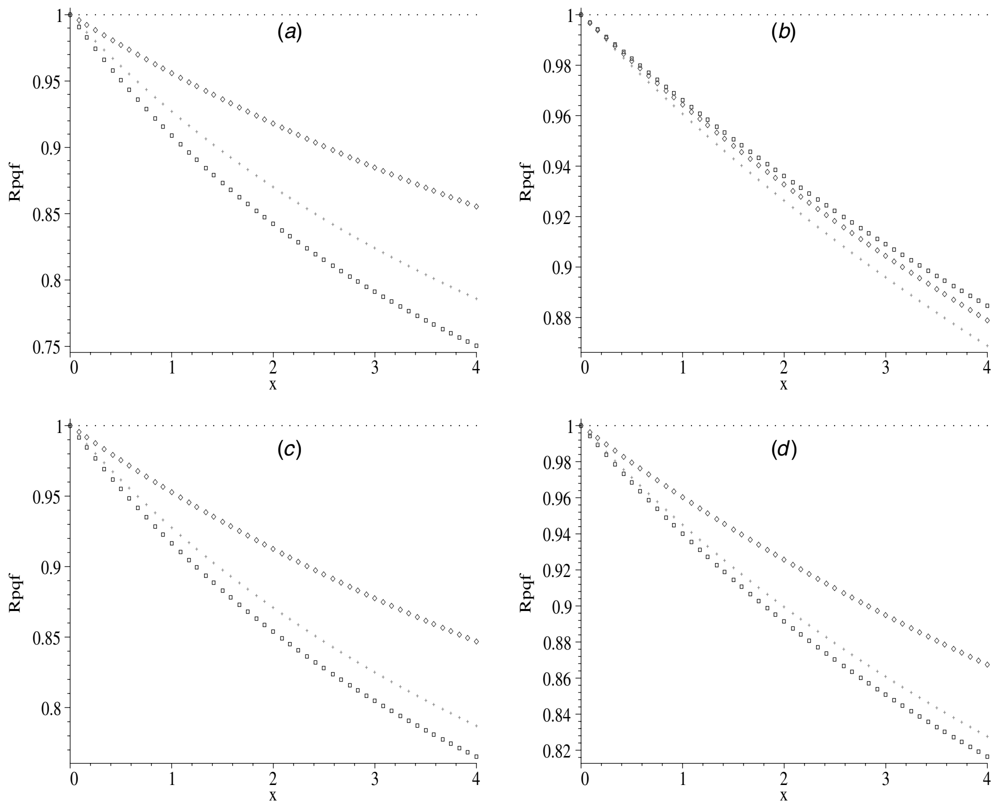

Figure 4. The variance ratio $R^{1,0}_{p,q,f}$ versus x for various (p, q) values: $(p, q) = (1, 1)$ (pointed line), $(p, q) = (1.01, 0.9)$ (diamond line), $(p, q) = (1.1, 0.8)$ (crossed line), $(p, q) = (1.2, 0.7)$ (boxed line).

where $\mathbf{S}^{(p',r)}$ is defined in (62). Since they are related to each other by the transformation $\mathrm{Re}(z) \leftrightarrow \mathrm{Im}(z)$, it is enough to study the former.

For $0 < z = \sqrt{x} << 1$, the ratio

$$R^{\mu,\nu}_{p,q,f}(x) = 2(\Delta X)^2 \tag{79}$$

of the variance $(\Delta X)^2$ in $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$ to the variance $1/2$ in the vacuum state behaves as

$$R^{\mu,\nu}_{p,q,f}(x) = 1 + \frac{2p^{\mu-1}}{f(p,q)q^{\nu-1}}\left[\left(\frac{2p^{\mu}}{q^{\nu-1}(pq+1)}\right)^{1/2} - 1\right] x + 0(x^2). \tag{80}$$

For $\mu = 1$ and $\nu = 0$,

$$R^{1,0}_{p,q,f}(x) = 1 + \frac{2q}{f(p,q)}\left(\sqrt{\frac{2pq}{1+pq}} - 1\right) x + o(x^2), \tag{81}$$

which shows that squeezing is present for small x values. Figure 4 allows us to conclude that there is a substantial squeezing increasing with x.

4.2.3. Quantum statistical properties of $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$: deformed photons. The deformed boson operators $b, b^{\dagger}$ may be interpreted as describing dressed photons, which may be invoked

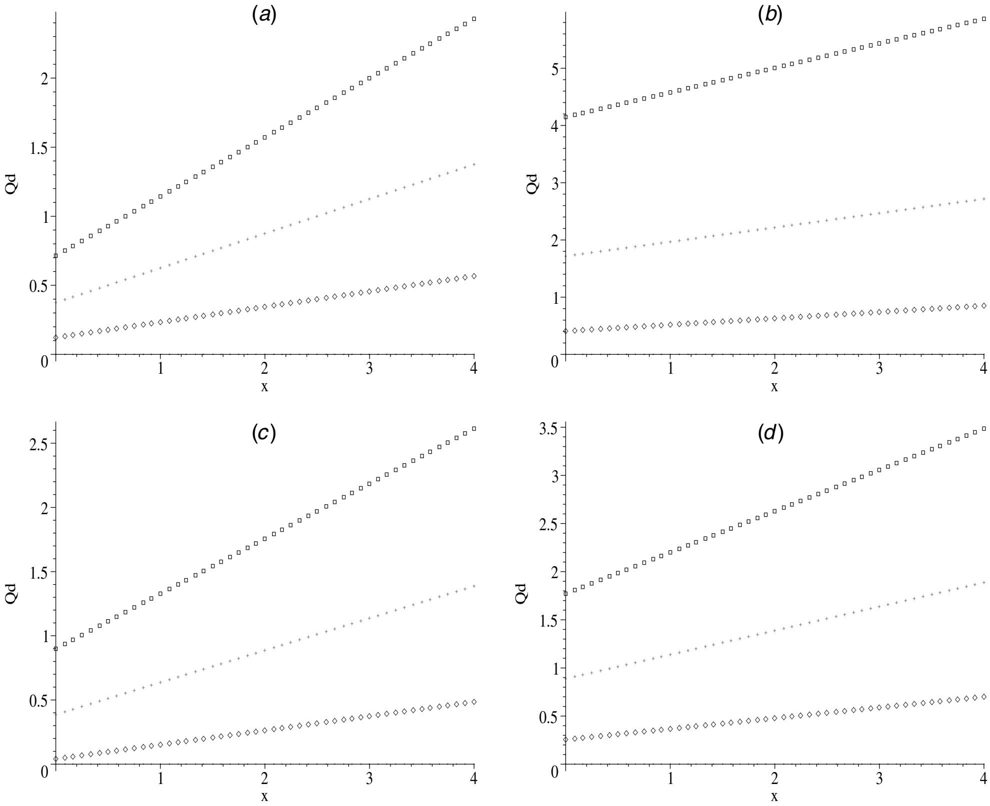

Figure 5. The Mandel parameter (84) versus x for various (p, q) values: $(p, q) = (1.01, 0.9)$ (diamond line), $(p, q) = (1.1, 0.8)$ (crossed line), $(p, q) = (1.2, 0.7)$ (boxed line).

in phenomenological models explaining some observable phenomena [43]. The physical applications previously considered for conventional photons may, therefore, be re-examined for those deformed photons.

(i) In such a context, analogous to the conventional case, we define the Mandel parameter as

$$Q_d(x) = \frac{((\Delta N)_d)^2 - \langle N\rangle_d}{\langle N\rangle_d}, \tag{82}$$

where

$$((\Delta N)_d)^2 = \langle N^2\rangle_d - (\langle N\rangle_d)^2 \quad \text{with} \quad \langle N\rangle_d = \langle b^\dagger b\rangle, \quad \langle N^2\rangle_d = \langle b^\dagger b b^\dagger b\rangle. \tag{83}$$

Namely,

$$Q_d(x) = \frac{1}{\mathcal{N}(x)} \sum_{n=0}^{+\infty} [n+1]^{\mu,\nu}_{p,q,f} \frac{x^n}{[n]^{\mu,\nu}_{p,q,f}!} - (x+1). \tag{84}$$

For $\mu = 1$ and $\nu = 0$ numerical calculations lead to figure 5, which shows that the Mandel parameter is positive and therefore corresponds to a super-Poissonian distribution.

(ii) To examine quadrature squeezing, we define the following Hermitian operators:

$$X_b = \frac{1}{\sqrt{2}}(b + b^\dagger), \qquad P_b = \frac{1}{\mathrm{i}\sqrt{2}}(b - b^\dagger). \tag{85}$$

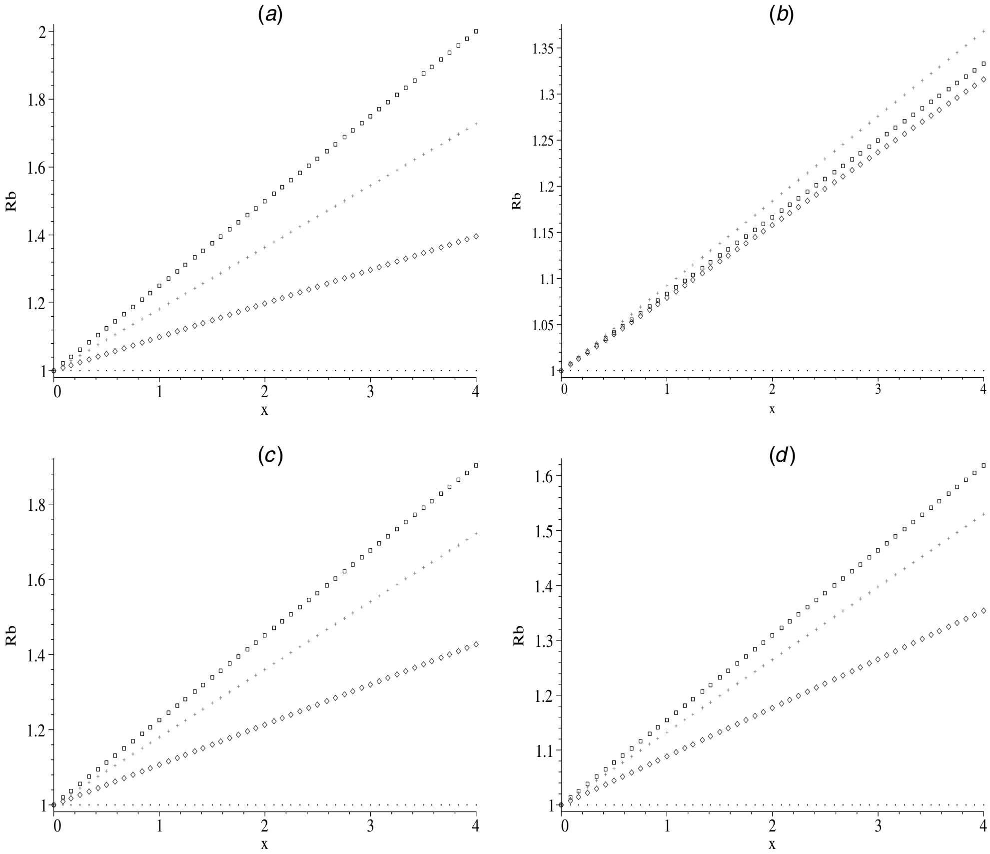

Figure 6. The variance ratio R_b versus x for various (p, q) values: $(p, q) = (1, 1)$ (pointed line), $(p, q) = (1.01, 0.9)$ (diamond line), $(p, q) = (1.1, 0.8)$ (crossed line), $(p, q) = (1.2, 0.7)$ (boxed line).

The quadrature variances $(\Delta X_b)^2$ and $(\Delta P_b)^2$ in any state satisfy the following uncertainty relation:

$$(\Delta X_b)^2(\Delta P_b)^2 \geqslant \tfrac{1}{4}|\langle[X_b, P_b]\rangle|^2. \tag{86}$$

For the deformed states $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$, it is straightforward to show that

$$\begin{aligned}(\Delta X_b)^2 = (\Delta P_b)^2 = \frac{1}{2}|\langle[X_b, P_b]\rangle| &= \frac{1}{2}\left(\langle[N+1]\rangle - \langle[N]\rangle\right)\\ &= \frac{1}{2}\left(\frac{1}{\mathcal{N}(x)}\sum_{n=0}^{+\infty}[n+1]^{\mu,\nu}_{p,q,f}\frac{x^n}{[n]^{\mu,\nu}_{p,q,f}!} - x\right),\end{aligned} \tag{87}$$

so

$$(\Delta X_b)^2(\Delta P_b)^2 = \tfrac{1}{4}|\langle[X_b, P_b]\rangle|^2. \tag{88}$$

Thus, it is found that the deformed states $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$ are intelligent states for the deformed operators X_b and P_b. As for the quadrature squeezing in $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$, let us define the ratio

$$R_b = \frac{2p^{\mu-1}}{q^{\nu-1}f(p,q)}(\Delta X_b)^2 \tag{89}$$

of $(\Delta X_b)^2$ to the variance $f(p,q)q^{\nu-1}/2p^{\mu-1}$ of X_b in the vacuum state.

For $\mu = 1$ and $\nu = 0$, figure 6 shows that R_b is always greater than 1; therefore, the states $|z\rangle^{1,0}_{p,q,f(p,q)}$ are not squeezed.

5. Concluding remarks

In this paper, we have defined a family of deformed states $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$, by introducing a generalized deformed number $[n]^{\mu,\nu}_{p,q,f} \equiv f(p,q)\frac{q^{\nu n}}{p^{\mu n}}\left(\frac{p^n - q^{-n}}{q - p^{-1}}\right)$. Relevant mathematical properties are investigated. In particular, for $\mu = 1$ and $\nu = 0$, the constructed deformed states reduce, up to a factor, to the coherent states discussed by Quesne, satisfying the mathematical statement of the unity resolution, with a measure not depending on the deformation parameters. Concrete examples of deformation functions $f(p,q)$ are provided and their influence on physical quantities is graphically illustrated.

Besides, the most significant physical characteristics of the states $|z\rangle^{\mu,\nu}_{p,q,f(p,q)}$ have been treated and analysed for particular values of $f(p,q)$. It has been found that, for conventional CS, described by the non-deformed operators a and $a^\dagger$, the states $|z\rangle^{1,0}_{p,q,f}$ exhibit a quantum squeezing either in quadrature X or in quadrature P and the geometry described by the Fubini–Study element is non-flat. On the other hand, for deformed operators described by the operators b and $b^\dagger$, the states $|z\rangle^{1,0}_{p,q,f}$ show the super-Poissonian statistics instead of the sub-Poissonian in the non-deformed case. Besides, they never show a squeezing either in X_b or in P_b.

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